General guidelines for marking theory papers

- Granularity for marks is 0.1 p for each tasks.
- Partial marks can be awarded for most aspects.
- A simple numerical error resulting from a typo is punished by 0.2 p unless the grading scheme explicitly says otherwise.
- Errors which cause dimensionally wrong results are punished by at least 50 % of the marks unless the grading scheme explicitly says otherwise.
- Propagating errors are not punished repeatedly unless they either lead to considerable simplifications or wrong results whose validity can easily be checked later (i.e. dimensional errors, divergence etc.).
- Correct equations originating from wrong physical ideas are not worth any points.
- All solutions should be graded according to only one marking scheme. If the student used multiple ideas/approaches, that marking scheme should be used which results in a higher score.

T1. Precession of the Earth's axis (10 pts)

Part A. The shape of the Earth (1.0 p)

A.1. Let us express the dimensions of $h_{\max}$, G, ω, M_E and R in terms of the base dimensions length L, mass M and time T:

$$[h_{\max}] = \mathrm{L},$$
$$[G] = \mathrm{L}^3\mathrm{M}^{-1}\mathrm{T}^{-2},$$
$$[\omega] = \mathrm{T}^{-1},$$
$$[M_E] = \mathrm{M},$$
$$[R] = \mathrm{L}.$$

The relation given in the problem should hold for the dimensions too:

$$\mathrm{L} = \left(\mathrm{L}^3\mathrm{M}^{-1}\mathrm{T}^{-2}\right)^{-1}\mathrm{T}^{-\beta}\mathrm{M}^{\gamma}\mathrm{L}^{\delta}.$$

After simplification we get:

$$\mathrm{L} = \mathrm{L}^{\delta-3}\mathrm{M}^{\gamma+1}\mathrm{T}^{2-\beta},$$

from which we get the following equations for the exponents:

$$0 = 2-\beta,$$
$$0 = \gamma+1,$$
$$1 = \delta-3.$$

From here we get $\beta = 2$, $\gamma = -1$ and $\delta = 4$.

Task A.1.	Pts
Expressing the dimension of G in terms of base dimensions	0.2
Setting up three equations for the exponents (0.1 p for each)	0.3
Correct values for exponents (0.1 p for each)	0.3
Total for Task A.1.	**0.8**

A.2. In the light of the result of the previous subpart the relation for $h_{\max}$ reads as

$$h_{\max} \propto \frac{\omega^2 R^4}{GM_E}.$$

Here $\omega = 2\pi/(24\ \mathrm{h}) = 7.27\times10^{-5}\ \mathrm{s}^{-1}$. Using 1 as the dimensionless constant, we get $h_{\max} = 21.9$ km.

Task A.2.	Pts
Correct calculation of ω (even if it was done inherently)	0.1
Correct value for $h_{\max}$	0.1
Total for Task A.2.	**0.2**

Part B. The time-averaged gravitational field of the Sun (3.2 p)

B.1. *Solution I: Using the gravitational potential.* At an arbitrary point on the z axis the gravitational potential $U(z)$ created by the ring is given by

$$U(z) = -G\frac{M_S}{\sqrt{z^2+d_{SE}^2}}.$$

The gravitational field can be found by differentiation with respect to z:

$$g_z(z) = -\frac{\mathrm{d}U}{\mathrm{d}z} = -GM_S\frac{z}{\left(z^2+d_{SE}^2\right)^{3/2}}.$$

Expanding this to first order in z we get:

$$g_z(z) \approx -\frac{GM_S}{d_{SE}^3}z.$$

The negative sign means that g_z points *towards the center* of the Sun ring.

Task B.1., Solution I.	Pts
Expressing the magnitude of $U(z)$ on the axis in terms of z correctly	0.2
Correct sign of $U(z)$	0.1
Expressing g_z as a derivative of $U(z)$. (0.1 p if negative sign is not included)	0.2
Calculating the derivative correctly	0.2
Approximate form of g_z for $\lvert z\rvert \ll d_{SE}$	0.1
Indicating correct direction in the figure	0.2
Total for Task B.1.	**1.0**

Solution II: Using the integration of fields. A small segment of the Sun ring with mass $\mathrm{d}M$ generates a field

$$\mathrm{d}g = \frac{G\,\mathrm{d}M}{z^2+d_{SE}^2}$$

on the symmetry axis of the ring at height z (see *Figure B.1*).

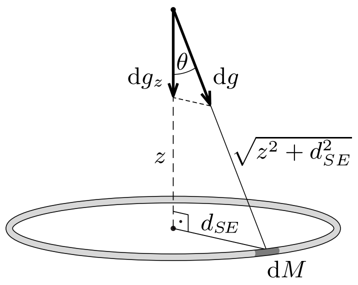


Figure B.1.

Due to symmetry, the net field at the same point is parallel with the axis, so only the corresponding component of this field should be taken:

$$\mathrm{d}g_z = -\mathrm{d}g \cos\theta,$$

where the negative sign indicates the $-z$ direction. The angle θ is the same for all segments of the ring and

$$\cos\theta = \frac{z}{\sqrt{z^2 + d_{SE}^2}}.$$

Using these three equations and integrating over the mass of the ring we get the net field on the axis at arbitrary position:

$$g_z = -GM_S \frac{z}{\left(z^2 + d_{SE}^2\right)^{3/2}}.$$

Using the relation $|z| \ll d_{SE}$ this simplifies to:

$$g_z \approx -GM_S \frac{z}{d_{SE}^3}.$$

Task B.1., Solution II.	**Pts**
Writing the gravitational field of an element of the ring	0.2
Figure with correct geometry	0.1
Taking only the z component for symmetry reasons	0.1
Summing/integrating over the whole ring	0.1
Calculating the g_z at arbitrary z correctly	0.2
Approximate form of g_z for $\|z\| \ll d_{SE}$	0.1
Indicating correct direction in the figure	0.2
Total for Task B.1.	**1.0**

B.2. *Solution I: Using Gauss's theorem.* The radial component of the field g_r in the plane of the Sun ring can be found from the gravitational Gauss's law (see *Figure B.2*).

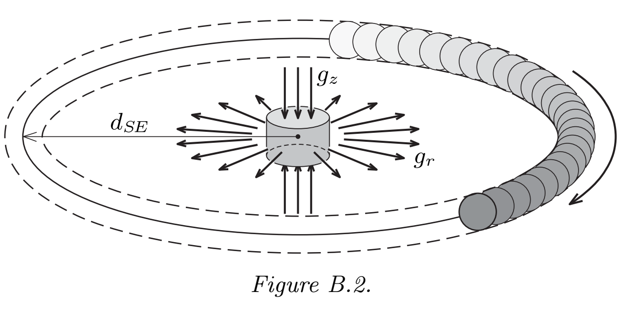


Figure B.2.

Apply Gauss's theorem for the cylindrical region of height $2|z|$ and radius r:

$$g_r\, 2z \times 2\pi r + g_z\, 2r^2\pi = 0,$$

from where we get

$$g_r(r) = -\frac{r}{2z} g_z(z) = \frac{GM_S}{2d_{SE}^3} r.$$

The field points radially *outwards*.

Task B.2., Solution I.	**Pts**
Idea of using Gauss's law	0.5
Taking a cylindrical Gaussian surface with axis z near the center of the Sun ring	0.4
Writing Gauss's law correctly in terms of radial and axial fields (0.3 p in case of mistake in areas, 0 p if the error is dimensional)	0.6
Final result for g_r is proportional to r (0 p if not)	0.3
Correct proportionality constant in g_r (0.1 p for error in prefactor, 0 p for dimensional error)	0.2
Indicating correct direction in the figure	0.2
Total for Task B.2.	**2.2**

Solution II: By integration of potential. Let us take a point P in the plane of the sung ring at distance r from the center (see *Figure B.3*).

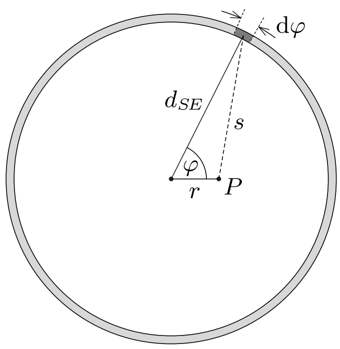


Figure B.3.

The distance s of a small element of the ring of angular size $\mathrm{d}\varphi$ located at angle φ with respect to point P is given by law of cosines:

$$s = \sqrt{d_{SE}^2 + r^2 - 2d_{SE} r \cos\varphi}.$$

The gravitational potential at point P due to the small segment can be written as

$$\mathrm{d}U = -\frac{GM_S}{s} \frac{\mathrm{d}\varphi}{2\pi},$$

so the net potential of the ring at point P is

$$U(r) = -\frac{GM_S}{2\pi} \int_0^{2\pi} \left(d_{SE}^2 + r^2 - 2d_{SE} r \cos\varphi\right)^{-\frac{1}{2}} \mathrm{d}\varphi.$$

Let us make the indegrand dimensionless:

$$U(r) = -\frac{GM_S}{2\pi d_{SE}} \int_0^{2\pi} \left(1 + \frac{r^2}{d_{SE}^2} - \frac{2r\cos\varphi}{d_{SE}}\right)^{-\frac{1}{2}} \mathrm{d}\varphi.$$

To simplify the integral we can use the fact that $r \ll d_{SE}$. Introducing the quantity

$$\varepsilon = \frac{r^2}{d_{SE}^2} - \frac{2r\cos\varphi}{d_{SE}}$$

($\varepsilon \ll 1$) we can expand the integrand up to second order in ε:

$$(1+\varepsilon)^{-\frac{1}{2}} \approx 1 - \frac{\varepsilon}{2} + \frac{3\varepsilon^2}{8}.$$

After writing back the expression of ε and keeping terms up to quadratic order in r/d_{SE} we get:

$$(1+\varepsilon)^{-\frac{1}{2}} \approx 1 - \frac{r^2}{2d_{SE}^2} + \frac{r\cos\varphi}{d_{SE}} + \frac{3r^2\cos^2\varphi}{2d_{SE}^2}.$$

The third term on the right side is canceled after integrating over φ, so the potential takes the form

$$U(r) = -\frac{GM_S}{2\pi d_{SE}} \int_0^{2\pi} \left(1 - \frac{r^2}{2d_{SE}^2} + \frac{3r^2\cos^2\varphi}{2d_{SE}^2}\right) \mathrm{d}\varphi.$$

Using that $\int_0^{2\pi} \cos^2\varphi \, \mathrm{d}\varphi = \pi$ (from the analogy with the calculation of real power in AC circuits), the integral can be evaluated:

$$U(r) = -\frac{GM_S}{2\pi d_{SE}} \left(2\pi - 2\pi\frac{r^2}{2d_{SE}^2} + \frac{3\pi r^2}{2d_{SE}^2}\right).$$

This simplifies to

$$U(r) = -\frac{GM_S}{d_{SE}} - \frac{GM_S r^2}{4d_{SE}^3}.$$

The gravitational field is the negative gradient of the potential:

$$g_r(r) = -\frac{\mathrm{d}U}{\mathrm{d}r} = \frac{GM_S}{2d_{SE}^3} r.$$

Task B.2., Solution II.	**Pts**
Expressing distance s from trigonometry	0.2
Writing the potential generated by a small element of the ring	0.1
Writing $U(r)$ as an integral	0.1
Taylor expansion of the integrand up to second order in r (0.1 p if only first order is calculated, 0.4 p if the term with $\cos^2\varphi$ is missing)	0.6
Integrating over φ (0.1 p if the term $\cos^2\varphi$ is missing)	0.2
Expressing g_z as a derivative of $U(z)$. (0.1 p if negative sign is not included)	0.2
Calculating the derivative correctly	0.1
Final result for g_r is proportional to r (0 p if not)	0.3
Correct proportionality constant in g_r (0.1 p for error in prefactor, 0 p for dimensional error)	0.2
Indicating correct direction in the figure	0.2
Total for Task B.2.	**2.2**

Part C. The torque acting on the Earth (2.6 p)

C.1. The ellipsoid of revolution can be transformed into a perfect sphere of radius R_e (see *Figure C.1.*) by stretching it uniformly along the polar diameter by a factor R_e/R_p, so the volume of the ellipsoid is given by

$$V_{\text{ellipsoid}} = \frac{4\pi}{3} R_e^3 \frac{R_p}{R_e} = \frac{4\pi}{3} R_e^2 R_p.$$

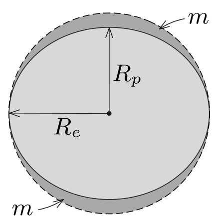


Figure C.1.

The volume of one of the excess regions is:

$$V = \frac{1}{2}\left(\frac{4\pi}{3}R_e^3 - \frac{4\pi}{3}R_e^2 R_p\right) = \frac{2\pi}{3} R_e^2 h_{\max}.$$

The density of the homogeneous Earth is $\varrho = 3M_E/(4\pi R_e^2 R_p)$, so the mass of one of the excess regions is the following:

$$m = \varrho V = \frac{3M_E}{4\pi R_e^2 R_p} \frac{2\pi}{3} R_e^2 h_{\max} = \frac{h_{\max}}{2R_p} M_E.$$

Task C.1.	**Pts**
Idea of stretching the ellipsoid into sphere	0.2
Volume of one of the excess regions	0.3
Correct expression for the density of Earth	0.1
Final result for m	0.2
Total for Task C.1.	**0.8**

C.2. The torque acting on the perfect sphere of radius R_e is zero due to symmetry. From the superposition principle outlined in the problem, it follows that the torque $\vec{\tau}$ acting on the ellipsoid-shaped Earth is equal in magnitude but opposite in direction to the torque $\vec{\tau'}$ acting on the two equivalent point masses (each of mass $2m/5$): $\vec{\tau} = -\vec{\tau'}$.

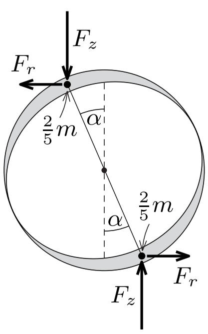


Figure C.2. The forces acting on the two point masses.

The magnitude of the torque acting on the point masses can be calculated with the help of *Figure C.2* as

$$|\vec{\tau}'| = |\vec{\tau}| = 2F_z R\sin\alpha + 2F_r R\cos\alpha,$$

where

$$F_z = \frac{2}{5}m|g_z| = \frac{2}{5}mGM_S\frac{R\cos\alpha}{d_{SE}^3},$$
$$F_r = \frac{2}{5}m|g_r| = \frac{2}{5}mGM_S\frac{R\sin\alpha}{2d_{SE}^3}.$$

Substituting these forces into the expression for τ' and simplifying we get:

$$|\vec{\tau}| = \frac{6}{5}\frac{GmM_S}{d_{SE}^3}R^2\sin\alpha\cos\alpha.$$

Using the result of part **C.1.** this can be written as

$$|\vec{\tau}| = \frac{3}{5}\frac{GM_E M_S}{d_{SE}^3}Rh_{\max}\sin\alpha\cos\alpha.$$

The torque $\vec{\tau}'$ is pointing out of the plane of *Figure C.2*, so the torque $\vec{\tau}$ acting on the ellipsoid-shaped Earth is pointing *into the plane*.

Task C.2.	**Pts**
Idea that the net torque acting on a perfect sphere is zero (even if it was done inherently)	0.1
Idea of $\vec{\tau} = -\vec{\tau}'$ (even if it was done inherently)	0.2
Including the terms coming from F_r and F_z in the torque correctly (0.4 p each)	0.8
Adding the two contributions with the correct sign	0.2
Calculation leading to the correct net torque	0.3
Correct direction for $\vec{\tau}$	0.2
Total for Task C.2.	**1.8**

Part D. Angular speed of the precession of the Earth's axis (2.0 p)

D.1. The torque acting on the Earth results a change in its angular momentum vector $\vec{L}$:

$$\vec{\tau} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t},$$

where $\vec{L}$ is parallel with the angular velocity of Earth's rotation and its magnitude (assuming a uniform mass distribution and neglecting the deviation from a sphere) is given by

$$|\vec{L}| = \frac{2}{5}M_E R^2\omega.$$

Since $\vec{\tau}$ (i.e. the rate of change of the angular momentum vector) is perpendicular to $\vec{L}$, the length of $\vec{L}$ remains constant but its direction changes, as shown in *Figure D.1.* As a result, the vector $\vec{L}$ sweeps along the side of a cone of half apex angle α.

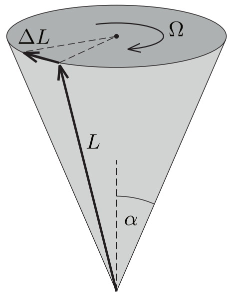


Figure D.1.

Drawing an analogy with a uniform circular motion, we can write an equation between L, its time derivative and the angular speed of precession:

$$\left|\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}\right| = \Omega_1|\vec{L}|\sin\alpha.$$

From this equation the angular speed of precession Ω_1 can be expressed:

$$\Omega_1 = \frac{\tau}{L\sin\alpha} = \frac{\frac{3}{5}GM_E M_S Rh_{\max}\sin\alpha\cos\alpha/d_{SE}^3}{\frac{2}{5}M_E R^2\omega\sin\alpha},$$

where we used our previous result for τ. After simplifying:

$$\Omega_1 = \frac{3}{2}\frac{GM_S h_{\max}}{d_{SE}^3 R\omega}\cos\alpha.$$

From this the period of precession:

$$T_1 = \frac{2\pi}{\Omega_1} = \frac{4\pi}{3}\frac{d_{SE}^3 R\omega}{GM_S h_{\max}\cos\alpha}.$$

Task D.1.	**Pts**
Newton's second law for rotational motion	0.2
Expressing the angular momentum in terms of ω and the moment of inertia	0.2
Writing the moment of inertia as $\frac{2}{5}M_E R^2$ (0.1 p for incorrect prefactor, 0 p for dimensional error)	0.2
Writing $\lvert\mathrm{d}\vec{L}/\mathrm{d}t\rvert$ in terms of L, Ω_1 and α	0.8
Using the equation $\Omega_1 = 2\pi/T_1$	0.1
Finding T_1 correctly	0.3
Total for Task D.1.	**1.8**

D.2. After substituting the data we get the numerical value of the period:

$$T_1 = 80\,600 \text{ years}.$$

Task D.2.	**Pts**
Correct numerical result for T_1. Full points for correct substitution into a dimensionally correct formula. 0 p if the substitution is incorrect or the formula has a dimensional error.	0.2
Total for Task D.2.	**0.2**

Part E. The effect of the Moon (1.2 p)

E.1. As we have seen in Part D, the angular speed of precession is proportional to the torque acting on the Earth, which is proportional to the quantity M_S/d_{SE}^3. If the effect of the Moon is taken into account, the torques exerted by the Sun and the Moon add up, and as a result

$$\Omega_2 = \frac{M_M/d_{ME}^3 + M_S/d_{SE}^3}{M_S/d_{SE}^3}\Omega_1.$$

From this we get a similar expression for the periods:

$$\frac{T_2}{T_1} = \frac{M_S/d_{SE}^3}{M_M/d_{ME}^3 + M_S/d_{SE}^3}.$$

Task E.1.	**Pts**
Stating that the torques of the Sun and the Moon add up	0.3
Calculating the torque exerted by the Moon or using that it is proportional to M_M/d_{SE}^3	0.4
Expressing T_2/T_1 correctly (0 p if $T_1 < T_2$)	0.3
Total for Task E.1.	**1.0**

E.2. After substitution we get

$$T_2 = 25\,400 \text{ years},$$

which is quite close to the value obtained by modern observations.

Task E.2.	**Pts**
Correct numerical result for T_2. 0 p if the result does not come from substitution (e.g. the student uses the value written in the introduction of the problem) or the substitution is incorrect. 0 p if the formula has dimensional error.	0.2
Total for Task E.2.	**0.2**

T2. Waves and Phase Transitions in Spin Systems (10.0 pts)

Part A. Precession and interactions of magnetic dipoles (1.2 points)

A.1: The angular momentum of the planar loop rotating around its perpendicular axis is given by

$$\vec{L} = MR^2\vec{\omega},$$

while the current generated by the rotation is $I = Q/T = \omega Q/2\pi$, then the magnetic dipole moment of the planar loop is given by

$$\vec{\mu} = IA\hat{\omega} = \frac{\omega Q}{2\pi}\pi R^2\hat{\omega} = \frac{Q}{2}R^2\vec{\omega}.$$

It follows that

$$\vec{\mu} = \frac{Q}{2M}\vec{L}$$

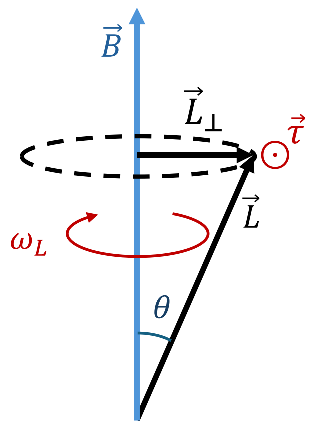
and

$$\gamma = \frac{Q}{2M}.$$

Grading scheme for Task A.1.	**Pts**
correct result for L (just magnitude is fine)	0.1
correct result for μ (just magnitude is fine)	0.1
correct result for γ	0.1
Total	**0.3**

A.2: The torque acting on a magnetic dipole due to an external magnetic field is given by

$$\vec{\tau} = \vec{\mu} \times \vec{B} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}.$$



Since only the component perpendicular to $\vec{B}$ can change, and it rotates with angular velocity ω_L, we write

$$\Rightarrow L\omega_L \sin\theta = -\mu B \sin\theta$$

from which we deduce

$$\omega_L = -\frac{\mu}{L}B = -\gamma B.$$

Note: The torque equation can be found by drawing an analogy to the electric dipole case.

Grading scheme for Task A.2.	**Pts**
Writing the torque formula	0.1
Realizing only the perpendicular component $L\sin\theta$ is changing	0.1
Correct magnitude for ω_L	0.1
Correct sign for ω_L	0.1
Total	**0.4**

Grading note: If the final answer for ω_L is written without justification, a maximum of 0.1 is given.

A.3: The magnetic field due to the first dipole on the second is

$$\vec{B}_{1 \text{ on } 2} = -\frac{\mu_0}{4\pi}\frac{\vec{\mu}_1}{d^3}.$$

Then

$$U = -\vec{\mu}_2 \cdot \vec{B}_{1 \text{ on } 2} = -\frac{\mu_0}{4\pi d^3}\mu_1\mu_2\cos(\pi - \theta)$$

leading to

$$U = \frac{\mu_0\gamma^2}{4\pi d^3}\vec{L}_1 \cdot \vec{L}_2$$

and

$$J_0 \equiv \frac{\mu_0\gamma^2}{4\pi d^3}.$$

Grading scheme for Task A.3.	**Pts**
Writing the interaction energy correctly	0.1
Writing the correct magnetic field magnitude	0.1
Writing the correct magnetic field direction	0.1
Correct magnitude for J_0	0.1
Correct sign for J_0	0.1
Total	**0.5**

Grading note: If the direction of magnetic field is not explicitly drawn or written, then credit is given if and only if the sign for interaction energy AND J_0 are correct.

Part B. Spin waves (4.5 points)

B.1: The ith spin interacts with the $i-1$ and $i+1$ spins, with an energy $E_i = -J(\vec{S}_{i-1} + \vec{S}_{i+1}) \cdot \vec{S}_i$ which is analogous to $E_i = -\vec{B}_i \cdot \vec{\mu}_i$. Using $\vec{\mu}_i = \gamma\vec{S}_i$, we find

$$\vec{B}_{i,\text{eff}} = \frac{J}{\gamma}\left(\vec{S}_{i-1} + \vec{S}_{i+1}\right).$$

Grading scheme for Task B.1.	**Pts**
Explicit understanding that $i-1$ and $i+1$ are the contributors for spin i	0.2
Correct result for effective magnetic field	0.1
Total	**0.3**

Grading note: Writing the correct result directly receives full credit.

B.2: Given the effective magnetic field

$$\frac{\mathrm{d}\vec{S}_i}{\mathrm{d}t} = \vec{\tau} = \vec{\mu}_i \times \vec{B}_{i,\mathrm{eff}}$$
$$= J\vec{S}_i \times \left(\vec{S}_{i-1} + \vec{S}_{i+1}\right).$$

Grading scheme for Task B.2.	Pts
Writing the rate of change of $\vec{S}$ using the effective magnetic field	0.1
Correct equation	0.2
Total	**0.3**

Grading note: No partial credit on this Task.

B.3: We can write the rate of change of the x and y components of $\vec{S}_i$, keeping in mind the approximation $S_{i,z} \approx S$ for all i.

$$\begin{aligned}\frac{\mathrm{d}S_{i,x}}{\mathrm{d}t} &= J\left[S_{i,y}\left(S_{i-1,z} + S_{i+1,z}\right)\right.\\ &\quad \left.-S_{i,z}\left(S_{i-1,y} + S_{i+1,y}\right)\right]\\ &\approx JS\left(2S_{i,y} - S_{i-1,y} - S_{i+1,y}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}S_{i,y}}{\mathrm{d}t} &= J\left[-S_{i,x}\left(S_{i-1,z} + S_{i+1,z}\right)\right.\\ &\quad \left.-S_{i,z}\left(S_{i-1,x} + S_{i+1,x}\right)\right]\\ &\approx -JS\left(2S_{i,x} - S_{i-1,x} - S_{i+1,x}\right).\end{aligned}$$

The structure of these equations along with the traveling wave behavior leads us to the ansatz

$$\begin{aligned}S_{i,x} &= \delta S\cos(kx - \omega t)\\ S_{i,y} &= \delta S\sin(kx - \omega t),\end{aligned}$$

Where δS is the amplitude. This yields

$$\begin{aligned}\omega\delta S\sin(kx - \omega t) = JS\delta S \cdot [&2\sin(kx - \omega t)\\ &- \sin(kx - \omega t - ka)\\ &- \sin(kx - \omega t + ka)].\end{aligned}$$

Using $\sin(A)+\sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$, we get

$$\omega(k) = 2JS\left[1 - \cos(ka)\right].$$

Note: We can show that the amplitude for S_x equals that of S_y, but substituting predictions with δS_x and δS_y, which leads to

$$\begin{aligned}\omega\delta S_x\sin(kx - \omega t) = JS\delta S_y \cdot [&2\sin(kx - \omega t)\\ &- \sin(kx - \omega t - ka)\\ &- \sin(kx - \omega t + ka)]\end{aligned}$$

$$\begin{aligned}\omega\delta S_y\cos(kx - \omega t) = JS\delta S_x \cdot [&2\cos(kx - \omega t)\\ &- \cos(kx - \omega t - ka)\\ &- \cos(kx - \omega t + ka)]\end{aligned}$$

which can only be satisfied given $\delta S_x = \delta S_y$.

Grading scheme for Task B.3.	Pts
Writing traveling waves as function of $kx \pm \omega t$ (either sign is okay, either trig function or complex exponentials is okay)	0.25
Same amplitude for S_x and S_y	0.25
Correct phase relation between S_x and S_y (a difference of $\pi/2$)	0.25
Writing the correct explicit equation of motion for either S_x or S_y	0.5
Explicit use of $S_z \approx S$	0.25
Correct final result ($\pm$ is okay)	0.5
Total	**2.0**

Grading note 1: credit should be given to the amplitudes equality even if not proved.
Grading note 2: In case the student does not arrive at the correct final solution, but has made a serious attempt at the algebra involving trig identities or complex exponentials, then up to 0.2 points may be rewarded.

B.4: For small k,

$$\omega(k) \approx 2JS\left[1 - 1 + \frac{1}{2}(ka)^2\right] = JSa^2k^2$$

The de Broglie relations are $E = \hbar\omega$ and $p = \hbar k$, plugging these into the expression for $\omega(k)$, we find

$$E = \hbar\omega = \frac{JSa^2}{\hbar}p^2 \equiv \frac{p^2}{2m_{\mathrm{eff}}},$$

where

$$m_{\mathrm{eff}} \equiv \frac{\hbar}{2JSa^2}.$$

Grading scheme for Task B.4.	Pts
Correct Taylor expansion result	0.2
Correct relation between momentum and wave vector	0.1
Correct relation between energy and angular frequency	0.1
Correct identification of effective mass	0.2
Total	**0.6**

Grading notes: If the student did not find the correct $\omega(k)$, but states that a massive particle has energy $E = p^2/2m$, then give 0.1 points (with a max of 0.3 given both de Broglie relations).

B.5: In this inelastic scattering, energy and momentum for the entire system (including the chain) is conserved. In particular, the spin wave does not have a momentum along the y-axis. Therefore

$$p_{\mathrm{in}}\cos\theta_{\mathrm{in}} = p_{\mathrm{out}}\cos\theta_{\mathrm{out}}.$$

Combining this with $E_n = p_n^2/2m_n$ valid for the neutron, we find

$$E_{\mathrm{out}} = \left(\frac{\cos\theta_{\mathrm{in}}}{\cos\theta_{\mathrm{out}}}\right)^2 E_{\mathrm{in}}$$

Using energy conservation, the energy E_s of the spin wave is $E_s = E_{\text{in}} - E_{\text{out}}$, so we get

$$E_s = \frac{\cos^2\theta_{\text{out}} - \cos^2\theta_{\text{in}}}{\cos^2\theta_{\text{out}}} E_{\text{in}}.$$

Conservation of momentum along the x-axis gives

$$\begin{aligned} p_s &= p_{\text{in}}\sin\theta_{\text{in}} - p_{\text{out}}\sin\theta_{\text{out}} \\ &= \sqrt{2m_nE_{\text{in}}}\left(\sin\theta_{\text{in}} - \frac{\cos\theta_{\text{in}}}{\cos\theta_{\text{out}}}\sin\theta_{\text{out}}\right). \end{aligned}$$

Then from $m_{\text{eff}} = p_s^2/2E_s$, we find

$$m_{\text{eff}} = \frac{\cos^2\theta_{\text{out}}\left(\sin\theta_{\text{in}} - \frac{\cos\theta_{\text{in}}}{\cos\theta_{\text{out}}}\sin\theta_{\text{out}}\right)^2}{\cos^2\theta_{\text{out}} - \cos^2\theta_{\text{in}}} m_n$$

which after simplifying gives

$$m_{\text{eff}} = \frac{\sin^2(\theta_{\text{in}} - \theta_{\text{out}})}{\cos^2\theta_{\text{out}} - \cos^2\theta_{\text{in}}} m_n = \frac{\sin(\theta_{\text{in}} - \theta_{\text{out}})}{\sin(\theta_{\text{in}} + \theta_{\text{out}})} m_n.$$

Grading scheme for Task B.5.	**Pts**
Conservation of momentum along y-axis	0.4
Conservation of momentum along x-axis	0.3
Conservation of energy	0.2
Relation between E_{out} and E_{in}	0.1
Correct final answer (should be in either two forms at the end, otherwise 0.2)	0.3
Total	**1.3**

Part C. Phase transitions in spin chains (4.3 points)

C.1: In a Boltzmann distribution, given the system's temperature T, the probability to find a system in a given state with energy ε_i is

$$p_i \propto \exp\left(-\frac{\varepsilon_i}{k_BT}\right).$$

In this case, the probability to find a spin up state of energy $\varepsilon_\uparrow = -hs_\uparrow = -h$ is

$$p_\uparrow \sim e^{h/k_BT}.$$

Therefore,

$$\frac{p_\uparrow}{p_\downarrow} = \frac{e^{h/k_BT}}{e^{-h/k_BT}} = e^{2h/k_BT}.$$

We also note that, normalization requires, $p_\uparrow + p_\downarrow = 1$. Thus, a full solution for $p_\uparrow$ and $p_\downarrow$ is

$$p_\uparrow = \frac{e^{h/k_BT}}{e^{h/k_BT} + e^{-h/k_BT}}, \tag{1}$$

$$p_\downarrow = \frac{e^{-h/k_BT}}{e^{h/k_BT} + e^{-h/k_BT}}. \tag{2}$$

Grading scheme for Task C.1.	**Pts**
Use of Boltzmann factor	0.2
Correct Boltzmann factor for up and down (0.1 each)	0.2
Correct final result for the ratio	0.1
Total	**0.5**

Grading note 1: if the ratio is written immediately without writing the Boltzmann factors separately then give the 0.2 associated with that.

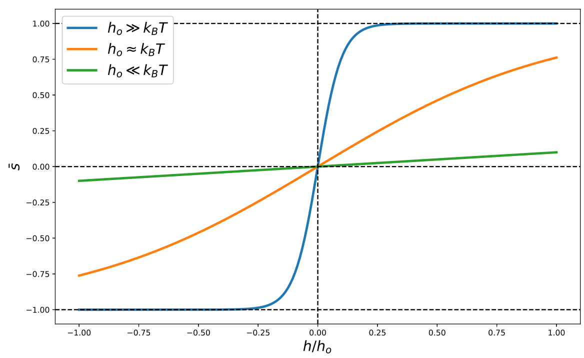
Grading note 2: even if the final answer is written immediately, the student must write $p \propto \exp(-E/k_BT)$ to receive full credit.

C.2: The average polarization of the system $\bar{s}$ can be written as

$$\begin{aligned} \bar{s} &= \frac{1}{N}\sum_i s_i, \\ &= \frac{1}{N}[Np_\uparrow \cdot 1 + Np_\downarrow \cdot (-1)], \\ &= p_\uparrow - p_\downarrow, \end{aligned}$$

where $Np_\uparrow$ and $Np_\downarrow$ are the number of spin vectors pointing up and down, respectively. Substituting for probabilities, we find

$$\bar{s} = \frac{e^{h/k_BT} - e^{-h/k_BT}}{e^{h/k_BT} + e^{-h/k_BT}} = \tanh\left(\frac{h}{k_BT}\right). \tag{3}$$



Grading scheme for Task C.2.	**Pts**
Deducing $\bar{s} = p_\uparrow - p_\downarrow$	0.2
Normalization condition $p_\uparrow + p_\downarrow = 1$	0.1
Final result for $\bar{s}$	0.1
Correct sketches (0.2 each)	0.6
Total	**1.0**

Grading note: For a sketch to be correct, it has to include labels of both axes and has intercept at $(0, 0)$. For $h_o \gg k_BT$, $\bar{s}$ has to approach ±1 quickly. For $h_o \ll k_BT$, it should look like a line with tiny slope. For $h_o \approx k_BT$, it should be in-between the two cases.

C.3: The energy of the system is minimized when all the spins are aligned, so

$$E_g = -\tilde{J}\sum_i 1 = -\tilde{J}(N-1) \approx -\tilde{J}N, \tag{4}$$

where we used $N \gg 1$.

Grading scheme for Task C.3.	Pts
Realizing the spins minimize their energy by aligning along the same direction	0.1
Correct result (both $N-1$ and N are fine)	0.1
Total	**0.2**

C.4:

$$E = -\tilde{J}\sum_i s_i s_{i+1} = -\tilde{J}\sum_i s_i \bar{s} = -\tilde{J}_{\text{eff}}\sum_i s_i,$$

where we define $\tilde{J}_{\text{eff}} = \tilde{J}\bar{s}$. Using $2\bar{s}$ is double counting the energy.

Grading scheme for Task C.4.	Pts
Realizing s_{i+1} can be replaced with $\bar{s}$	0.1
Correct final result	0.1
Total	**0.2**

Grading note: If the student uses $2\bar{s}$, then no partial credit. However, no propagating error from this specific mistake for the following parts.

C.5: By looking at the earlier equation given in part C.4, and comparing it to what we had earlier in C.2, we find that the average polarization $\bar{s}$ should satisfy the following transcendental equation:

$$\bar{s} = \tanh\left(\frac{\tilde{J}_{\text{eff}}}{k_B T}\right) = \tanh\left(\frac{\tilde{J}\bar{s}}{k_B T}\right).$$

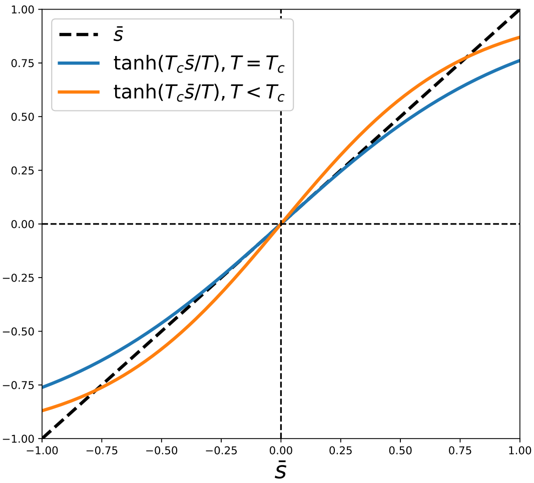
With the help of the plots we did earlier in C.2, one can see that for $\tilde{J} \ll k_B T$, there exists only one simple solution, $\bar{s} = 0$. Where as for $\tilde{J} \gg k_B T$, there exist two non-trivial solutions. As clarified by the figure below, this crossing behavior occurs when the tangent line for $\tanh\left(\tilde{J}\bar{s}/k_B T\right)$ at $\bar{s} = 0$ equals the slope of $\bar{s}$:

$$\frac{d}{d\bar{s}}\tanh\left(\frac{\tilde{J}\bar{s}}{k_B T_c}\right)\bigg|_{\bar{s}=0} = \frac{d}{d\bar{s}}\bar{s}\bigg|_{\bar{s}=0}$$

$$\Rightarrow \frac{\tilde{J}}{k_B T_c}\,\frac{1}{\cosh^2\left(\tilde{J}\bar{s}/k_B T_c\right)}\bigg|_{\bar{s}=0} = 1$$

$$\Rightarrow \frac{\tilde{J}}{k_B T_c} = 1,$$

leading to $T_c = \tilde{J}/k_B$.



Grading scheme for Task C.5.	Pts
stating $\bar{s} = \tanh\left(\tilde{J}_{\text{eff}}/k_B T\right)$	0.1
Replacing $\tilde{J}_{\text{eff}}$ for h in $\bar{s}$ results from C.2	0.2
Realizing that there is one trivial solution for $\tilde{J} \ll k_B T$	0.2
Realizing that there are two non-trivial solutions for $\tilde{J} \gg k_B T$	0.2
Clearly stating the condition at when number of solutions changes	0.3
Correct final result for T_c	0.2
Total	**1.2**

C.6: Near the critical temperature, the average polarization is small. Thus, we can approximate the transcendental equation as

$$\bar{s} = \tanh\left(\frac{T_c}{T}\bar{s}\right) \approx \frac{T_c}{T}\bar{s} - \frac{1}{3}\left(\frac{T_c}{T}\bar{s}\right)^3.$$

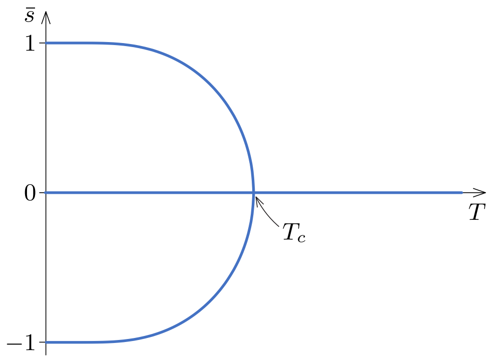
We see that $\bar{s} = 0$ is still a solution. After rearranging for $\bar{s} \neq 0$, we get

$$\bar{s} = \pm\sqrt{3\left[\left(\frac{T}{T_c}\right)^2 - \left(\frac{T}{T_c}\right)^3\right]}$$

$$= \pm\sqrt{3\left(\frac{T}{T_c}\right)^2\left(1 - \frac{T}{T_c}\right)}$$

$$\approx \pm\sqrt{3\frac{T_c - T}{T_c}},$$

where we have used $(T/T_c)^2 \approx 1$.



Grading scheme for Task C.6.	Pts
Using the proper approximation for tanh	0.1
Correct non-trivial solutions for $\bar{s}$, even if not fully simplified (0.1 each)	0.2
Sketch $\bar{s} = 0$ as the only solution above T_c	0.1
The non trivial solutions are vertical at T_c	0.2
$\bar{s} = 0$ is sketched as a solution for $T < T_c$	0.1
The two non-trivial solutions monotonically increase to ± 1 at $T = 0$ (0.1 each)	0.2
Either non-trivial solutions have a zero slope approaching $T = 0$.	0.1
Total	**1.0**

Grading note 1:The vertical slope at T_c has to be very clear. It is an important physical signature of phase transitions, so the student has to emphasize it in the sketch to receive credit for it.

Grading note 2: the credit associated with slope approaching zero as $T \to 0$ can be given even if the student does not plot the negative solution.

C.7: In the absence of magnetic fields, when $T > T_c$, there is only one solution at $\bar{s} = 0$, which means that the system cannot maintain a net polarization. However, applying a magnetic field leads to a net magnetization, which is a characteristic of **paramagnets**. On the other hand, when $T < T_c$, the system can support a net magnetization even in the absence of magnetic fields, which is the characteristic of **ferromagnets**.

Grading scheme for Task C.7.	Pts
Correct choice for $T > T_c$	0.1
Correct choice for $T < T_c$	0.1
Total	**0.2**

Grading note: No justification is needed for credit. If multiple choices are chosen, then no credit. In case the choices were changed, only give credit if the final choice is correct and clear.

T3. Physics of the Atmosphere (10 pts)

Part A. Surface temperature of the Earth (1.2 points)

A.1: The cross-section area receiving the solar radiation (falling as parallel rays) is πR_E^2, so taking into account the absorbed portion is a fraction $1-a$ of the total incident radiation, we find

$$P_0 = (1-a)\pi R_E^2 F_s.$$

Grading scheme for Task A.1.	Pts
Correct effective cross section area $A = \pi R_E^2$	0.1
Correct final answer	0.1
Total	**0.2**

Grading note: If the student uses a different cross sectional area, only 0.1 is given, provided it is the only mistake.

A.2: A black body radiates according to the Stefan-Boltzmann law, $P_{\mathrm{bd}} = \sigma A T^4$, where σ is the Stefan-Boltzmann constant and A is the total surface area of the black body. At steady state

$$P_{\mathrm{bd}} = P_0 \Rightarrow \sigma(4\pi R_E^2)T_{g0}^4 = (1-a)\pi R_E^2 F_s$$

$$\Rightarrow T_{g0} = \left((1-a)\frac{F_s}{4\sigma}\right)^{1/4} \approx 255\,\mathrm{K} \approx -18\,^\circ\mathrm{C}.$$

Grading scheme for Task A.2.	Pts
Energy balance	0.1
Correct explicit blackbody radiation formula, using the surface area of a sphere	0.1
Correct numerical value	0.1
Total	**0.3**

A.3: In the presence of the atmospheric layer, we write down the energy transfer balance in two regions: between the Earth's surface and the atmosphere, and between the atmosphere and outer space. Let the power radiated from Earth be P_E and the power radiated from each side of the atmosphere be P_{atmo}, then

$$P_E = P_{\mathrm{atmo}} + t_{\mathrm{sw}} P_0,$$
$$t_{\mathrm{lw}} P_E + P_{\mathrm{atmo}} = P_0.$$

Solving this system of equations and using $P_E = \sigma(4\pi R_E^2)T_g$ we find

$$T_g = \left(\frac{1+t_{\mathrm{sw}}}{1+t_{\mathrm{lw}}}\right)^{1/4} T_{g0} \approx 286\,\mathrm{K} \approx 13\,^\circ\mathrm{C}.$$

Grading scheme for Task A.3.	Pts
Statement on radiation balance in the region outside the atmosphere	0.1
Statement on radiation balance in the region between the atmosphere and Earth	0.2
Using t_{sw} correctly	0.1
Using t_{lw} correctly	0.1
Correct numerical result (if only analytical, then only 0.1)	0.2
Total	**0.7**

Part B. The absorption spectrum of atmospheric gases (1.8 points)

B.1: Let the natural (unstretched) length of the spring be l_0 and let x_A, x_B be the positions of particles A and B, respectively. The equation of motion of each particle due to the spring force can be written as:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_A = +\frac{k}{m_A}(\ell-\ell_0),$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_B = -\frac{k}{m_B}(\ell-\ell_0),$$

where $\ell = x_B - x_A$ is the instantaneous length of the spring. Taking the difference of the two equations gives

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ell = -k\left(\frac{1}{m_B}+\frac{1}{m_B}\right)(\ell-\ell_0).$$

This is the equation of motion of a single effective particle attached to a spring with a spring constant k and an effective mass or reduced mass μ, given by:

$$\mu = \frac{1}{\frac{1}{m_A}+\frac{1}{m_B}} = \frac{m_A m_B}{m_A+m_B}.$$

Thus, the system undergoes a simple harmonic motion with an angular frequency:

$$\omega_d = \sqrt{\frac{k}{\mu}} = \sqrt{k\frac{m_A+m_B}{m_A m_B}}.$$

Grading scheme for Task B.1.	Pts
Writing down correct equations of motion for A and B (0.1 each)	0.2
Studying the equation of motion for x_A-x_B	0.1
Correct answer	0.2
Total	**0.5**

Grading note: A maximum of 0.2 points are given if the correct result is cited without justification.

B.2: The difference in energy between two consecutive levels in a quantum harmonic oscillator is given by $\hbar\omega$. So the energy of the photon is given by

$$E = \hbar\omega_d.$$

Grading scheme for Task B.2.	Pts
Correct result (Give 0.1 if h is used instead of $\hbar$. No other numerical factors receive credit.)	0.2
Total	**0.2**

B.3: The observed shift in the spectral line from f_0 is due to the Doppler effect. When the source is moving towards the observer with velocity v the frequency is shifted according to

$$f = f_0 \left(1 + v/c\right).$$

Thus, the shift in frequency is given by:

$$f - f_0 = \frac{v}{c} f_0.$$

Grading scheme for Task B.3.	**Pts**
Writing down an expression for Doppler effect (even if incorrect)	0.1
Correct answer	0.1
Total	**0.2**

B.4: To find the normalization constant C, we require that the total probability is equal to 1. This leads to:

$$\int_{-\infty}^{\infty} p(v)\, \mathrm{d}v = 1 \Rightarrow C \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{mv^2}{2k_BT}}\, \mathrm{d}v.$$

Using the integration formula provided, with $x = v$ and $a = \frac{m}{k_BT}$ we obtain:

$$C = \sqrt{\frac{m}{2\pi k_BT}}.$$

Grading scheme for Task B.4.	**Pts**
Normalization condition (even if done incorrectly from 0 to ∞)	0.1
Correct result	0.1
Total	**0.2**

B.5: Using the result of **B.4**, we obtain the following expression for the speed of a molecule in terms of the frequencies f and f_0:

$$v = \frac{f - f_0}{f_0} c.$$

We plug this back into the probability distribution formula to obtain:

$$p(f) \propto \exp\left[-\frac{mc^2}{2k_BT}\left(\frac{f - f_0}{f_0}\right)^2\right].$$

This gives the probability distribution for observing a molecule whose spectral line is Doppler shifted from f_0 to f.

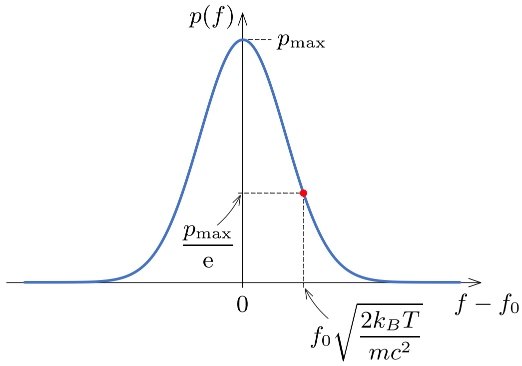
Grading scheme for Task B.5.	**Pts**
Replacing v by the Doppler effect result	0.1
Correct exponential dependence	0.2
Total	**0.3**

Grading note: If the student uses an incorrect Doppler effect formula, but one that matches their attempt in **B.3**, they get the 0.1 points.

B.6: The probability distribution $p(f)$ follows a Gaussian profile in the frequency shift $f - f_0$. The center of the profile is 0 and it drops to $1/e$ of its maximum value when the argument of the exponential is -1. This happens when

$$f^* - f_0 = f_0\sqrt{\frac{2k_BT}{mc^2}}.$$

The shape of the distribution can be seen in the figure below.



Grading scheme for Task B.6.	**Pts**
The distribution is has a single peak at zero	0.1
The distribution is symmetric	0.1
The distribution decays to zero on both ends	0.1
$f^* - f_0$ is correct	0.1
Total	**0.4**

Part C. Stability of air in the atmosphere (2.7 points)

C.1: Consider a thin horizontal layer of thickness dz and surface area S. Since the air is in hydrostatic equilibrium, it's weight must be balanced by the difference in pressure forces. This results in the following relation:

$$p(z)S = p(z + \mathrm{d}z)S + \rho(z)gS\mathrm{d}z.$$

Simplifying and rearranging terms gives:

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho(z)g$$

The negative sign indicates a decrease in pressure with hight as expected.

Grading scheme for Task C.1.	**Pts**
Sum of forces equals zero	0.1
Correct pressure force above and below	0.1
Correct final answer	0.1
Total	**0.3**

C.2: Assuming we can treat air as an ideal gas, we can use the ideal gas law to express the density of air in terms of its pressure and temperature

$$pV = nRT \Rightarrow p(z)V = \frac{m}{\mu_{\text{air}}} RT(z).$$

Rewriting this in terms of the density gives:

$$\rho(z) = \frac{p(z)\mu_{\text{air}}}{RT(z)}.$$

Now we substitute the density expression into the expression obtained in **C.1**. This gives:

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\mu_{\text{air}} p(z)}{RT(z)} g$$

Grading scheme for Task C.2.	**Pts**
Ideal gas law	0.1
Correct final answer	0.1
Total	**0.2**

C.3: Assuming an isothermal atmosphere (i.e., constant temperature with altitude), $T(z) = T$, the equation simplifies to:

$$\frac{\mathrm{d}p}{p} = -\frac{\mu_{\text{air}}}{RT} g\mathrm{d}z.$$

Integrating both sides and assuming the pressure at height 0 is p_0 leads to:

$$\ln\left[\frac{p(z)}{p_0}\right] = -\frac{\mu_{\text{air}}}{RT} gz.$$

In a different form:

$$p(z) = p_0 \exp\left(-\frac{\mu_{\text{air}}}{RT} gz\right).$$

Grading scheme for Task C.3.	**Pts**
Recognizing a separable differential equation	0.1
Correct final answer	0.1
Total	**0.2**

C.4: Since the small mass of air is displaced adiabatically, it must satisfy the adiabatic condition for an ideal gas:

$$pV^{\gamma} = \text{const.},$$

where $\gamma = c_p/c_V$ is the adiabatic index, and c_p, c_V are the molar specific heats at constant pressure and volume respectively. Writing the volume in terms temperature and pressure using the ideal gas law gives:

$$p(T/p)^{\gamma} = \text{const.} \quad \Rightarrow \quad p^{1-\gamma}T^{\gamma} = \text{const.}$$

Taking the derivative of this expression with respect to the height z, we obtain:

$$(1-\gamma)p^{-\gamma}\frac{\mathrm{d}p}{\mathrm{d}z}T^{\gamma} + \gamma p^{1-\gamma}T^{\gamma-1}\frac{\mathrm{d}T}{\mathrm{d}z} = 0.$$

Simplifying and rearranging to have an expression for the adiabatic lapse rate gives:

$$\frac{\mathrm{d}T}{\mathrm{d}z} = -\frac{1-\gamma}{\gamma}\frac{T(z)}{p(z)}\frac{\mathrm{d}p}{\mathrm{d}z}.$$

We now substitute the hydrostatic pressure gradient obtained in **C.3** to get:

$$\frac{\mathrm{d}T}{\mathrm{d}z} = -\frac{1-\gamma}{\gamma}\frac{T(z)}{p(z)}\left[-\frac{p(z)\mu_{\text{air}}}{RT(z)}g\right] = \frac{1-\gamma}{\gamma}\frac{\mu_{\text{air}}}{R}g.$$

But $\gamma = c_p/c_V$, so

$$\frac{\mathrm{d}T}{\mathrm{d}z} = \frac{1-c_p/c_V}{c_p/c_V}\frac{\mu_{\text{air}}}{R}g = -\frac{\mu_{\text{air}}}{c_p}g,$$

where we used $c_p - c_V = R$.

This expression for the adiabatic lapse rate shows that the temperature drops linearly with height in an adiabatic atmosphere.

Grading scheme for Task C.4.	**Pts**
Writing the adiabatic relation in any form	0.1
Relating dT/dz to dp/dz	0.3
Correct final result	0.2
Total	**0.6**

C.4: To find the angular frequency of small oscillations of the air parcel, we begin by applying Newton's second law, where the primary forces acting on the parcel are buoyancy and gravity.

$$\delta m\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \rho_a(z)g\delta V - \delta m g,$$

where δm is the mass of the air parcel, δV is its volume and ρ_a is the density of the surrounding air. We can express the mass of the parcel in terms of its density ρ_p as $\delta m = \rho_p\delta V$. Substituting and simplifying gives:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \frac{\rho_a(z+\delta z) - \rho_p(z+\delta z)}{\rho_p(z+\delta z)}g.$$

Assuming the parcel is at the same pressure as the atmosphere at $z + \delta z$, the density can be expressed in terms of temperature using the ideal gas law $\rho \propto 1/T$. This allows us to rewrite the last expression as:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \frac{T_p(z+\delta z) - T_a(z+\delta z)}{T_a(z+\delta z)}g.$$

We can now express the temperature at $z+\delta z$ in terms of the lapse rates and the temperature at z using the definition $T(z+\delta z) = T(z) + \Gamma\delta z$. Therefore:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \frac{T(z) + \Gamma\delta z - T(z) - \Gamma_a\delta z}{T(z) + \Gamma_a\delta z}g.$$

Simplifying the numerator and neglecting the infinitesimal term $\Gamma_a\delta z$ in the denominator gives:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \frac{\Gamma - \Gamma_a}{T}g\delta z.$$

This is the equation of a simple harmonic oscillator, where the angular frequency is given by:

$$\omega = \sqrt{\frac{\Gamma_a - \Gamma}{T}g} = \sqrt{\frac{\mu_{\text{air}}g/c_p - \Gamma}{T}g}$$

The motion is stable whenever $\Gamma_a = \mu_{\text{air}}g/c_p > \Gamma$.

Grading scheme for Task C.5.	Pts
Inclusion of gravitational force with parcel density	0.2
inclusion of buoyancy force with air density	0.3
Correct equation of motion	0.2
Relating density to inverse temperature	0.2
Using appropriate approximation	0.2
Correct stability requirements	0.1
Correct angular frequency of small oscillation	0.2
Total	**1.4**

Part D. Moisture (2.7 points)

D.1: The change of entropy across a phase transition (evaporation in this case) is related to the latent heat of evaporation. If there was a mass m of liquid water, then $Q_{\text{evaporation}} = Lm$, then

$$\Delta S = \frac{Lm}{T}.$$

It is known that the volume of vapor is significantly larger than the volume of liquid of the same mass, therefore $\Delta V \approx V_{\text{vapor}}$, which can be found using the ideal gas law

$$V_{\text{vapor}} = \frac{nRT}{p_s(T)}.$$

The mass can be related to the number of moles n via $m = \mu_{H_2O} n$, then

$$\frac{\mathrm{d}p_s}{\mathrm{d}T} = \frac{\mu_{H_2O} L p_s}{RT^2}.$$

Grading scheme for Task D.1.	Pts
Correct entropy change	0.2
$V_{\text{vapor}} \gg V_{\text{liquid}}$	0.2
Correct final answer	0.1
Total	**0.5**

D.2: We can integrate the relationship found in **D.1** by separating variables to find

$$\ln\left[\frac{p_s(T)}{p_{so}}\right] = -\frac{\mu_{H_2O} L}{R}\left(\frac{1}{T} - \frac{1}{T_o}\right).$$

Note that L is strictly a function of temperature, but we are assuming that L is a constant for the range of temperatures we investigate. Rearranging, we find

$$p_s(T) = p_{so} \exp\left[-\frac{\mu_{H_2O} L}{R}\left(\frac{1}{T} - \frac{1}{T_o}\right)\right].$$

Grading scheme for Task D.2.	Pts
Recognizing a separable differential equation	0.1
Correct final answer	0.1
Total	**0.2**

D.3: Formation of liquid water happens when the partial pressure of water inside the parcel reaches the saturation pressure at a given temperature. The partial pressure of water vapor p_w can be related to the total pressure of the parcel p as

$$p_w = \frac{n_{H_2O}}{n_{\text{air}}} p = \frac{m_{H_2O}/\mu_{H_2O}}{m_{\text{air}}/\mu_{\text{air}}} p = \phi \frac{\mu_{\text{air}}}{\mu_{H_2O}} p.$$

Given that the air parcel is rising adiabatically, $p^{1-\gamma} T^{\gamma} = \text{const.}$, so

$$p(T) = p_i \left(\frac{T}{T_i}\right)^{c_p/R}.$$

Therefore, the transcendental equation that we need to solve is

$$\phi \frac{\mu_{\text{air}}}{\mu_{H_2O}} p_i \left(\frac{T_l}{T_i}\right)^{\frac{c_p}{R}} = p_{so} \exp\left[-\frac{\mu_{H_2O} L}{R}\left(\frac{1}{T_l} - \frac{1}{T_i}\right)\right].$$

This can be rearranged to get

$$T_l = \frac{1}{\frac{1}{T_i} - \frac{R}{\mu_{H_2O} L} \ln\left[\phi \frac{\mu_{\text{air}}}{\mu_{H_2O}} \frac{p_i}{p_{so}} \left(\frac{T_l}{T_i}\right)^{c_p/R}\right]}.$$

Substituting the numerical values, we get

$$T_l = \frac{1000\text{ K}}{3.481 - 0.4695 \ln\left(\frac{T_l}{290.15\text{ K}}\right)}.$$

Solving this iteratively, we find $T \approx 286.8\,\mathrm{K} \approx 13.7\,°C$.

Grading scheme for Task D.3.	Pts
Using Dalton's law	0.4
correctly relating the moles ratio to mass ratio	0.2
Stating $p(T)$ for an adiabatic process	0.1
Understanding that partial pressure of water needs to reach saturation for condensation to start	0.5
Attempting to perform iterative search for the solution of the transcendental equation (by isolating T on one side)	0.4
Correct numerical solution	0.4
Total	**2.0**

Grading note: At most 0.4 pts can be given if the student does not use the partial pressure of water but uses the total pressure of air parcel.

Part E. Sun halo (1.6 points)

E.1: Using the notations of *Figure E*, the total angle of deviation δ can be written as the sum of the deviations in the two refractions:

$$\delta = \alpha - \alpha' + \beta - \beta'.$$

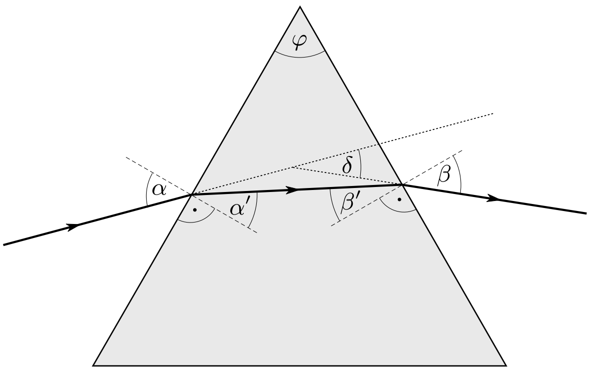


Figure E.

Consider the triangle of interior angles φ, $90° - \alpha'$ and $90° - \beta'$. Since the sum of these angles add up to 180°, we get

$$\varphi = \alpha' + \beta',$$

so δ simplifies to

$$\delta = \alpha + \beta - \varphi.$$

The relationship between α and α' (and similarly between β and β') is given by Snell's law:

$$\frac{\sin\alpha}{\sin\alpha'} = n, \qquad \frac{\sin\beta}{\sin\beta'} = n\,.$$

Expressing β in terms of α':

$$\sin\beta = n\sin\beta' = n\sin(\varphi - \alpha'),$$

From Snell's law α' can be written as

$$\alpha' = \arcsin\left(\frac{\sin\alpha}{n}\right).$$

Thus, β in terms of α is given by

$$\beta = \arcsin\left\{n\sin\left[\varphi - \arcsin\left(\frac{\sin\alpha}{n}\right)\right]\right\}.$$

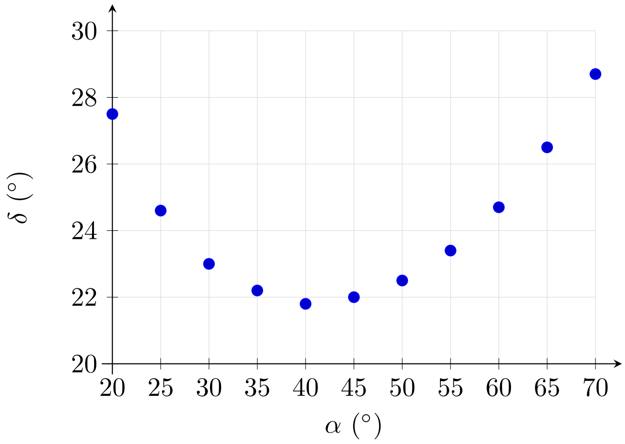
Finally, we get the result for δ:

$$\delta = \alpha + \arcsin\left\{n\sin\left[\varphi - \arcsin\left(\frac{\sin\alpha}{n}\right)\right]\right\} - \varphi.$$

Grading scheme for Task E.1.	**Pts**
Writing Snell's law for the two refractions (0.1 each)	0.2
Equation for δ in terms of α, β and φ	0.2
Using that $\alpha' + \beta' = \varphi$	0.1
Correct calculation leading to δ	0.2
Final formula for δ (any other equivalent form is acceptable)	0.1
Total	**0.8**

E.2: Notice that the situation corresponds to the case discussed in part **E.1** with $\varphi = 60°$. Here is the data table after substituting different values of α:

α	δ	α	δ
20°	27.5°	50°	22.5°
25°	24.6°	55°	23.4°
30°	23.0°	60°	24.7°
35°	22.2°	65°	26.5°
40°	21.8°	70°	28.7°
45°	22.0°	–	–



Grading scheme for Task E.2.	**Pts**
Substituting into the formula for δ correctly for all values of α (if at least 6 data points are calculated, 0.1 p can be given)	0.2
Data points are plotted in the correct graph	0.2
δ has a local minimum	0.2
Total	**0.6**

E.3: The minimum value of δ is around 21.8°, so that is the angle with respect to the direction of Sun where the halo appears.

Grading scheme for Task E.3.	**Pts**
Reading the minimal value of δ	0.1
Concluding that the angular size of halo corresponds to the minimal value of δ	0.1
Total	**0.2**